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L^1 estimates for dissipative wave equations in exterior domains ☆

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Abstract

We study the exterior initial–boundary value problem for the linear dissipative wave equation $(\Box + \partial_t)u = 0$ in $\Omega \times (0, \infty)$ with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ and $u|_{\partial\Omega} = 0$, where Ω is an exterior domain in N-dimensional Euclidean space $\mathbb{R}^N$. We first show higher local energy decay estimates of the solution $u(t)$, and then, using the cut-off technique together with those estimates, we can obtain the L^1 estimate of the solution $u(t)$ when $N \geqslant 3$, that is, $\|u(t)\|_{L^1(\Omega)} \leqslant C(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)} + \|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)})$ for $t \geqslant 0$, where $n = [N/2]$ is the integer part of $N/2$. Moreover, by induction argument, we derive the higher energy decay estimates of the solution $u(t)$ for $t \geqslant 0$.

Keywords: Dissipative wave equation; L^1 estimates; Decay; Exterior domain

1. Introduction and results

Let Ω be an exterior domain in N-dimensional Euclidean space $\mathbb{R}^N$ for $N \geqslant 2$ with a smooth boundary $\partial\Omega$ and its complement $\Omega^c = \mathbb{R}^N \setminus \Omega$ will be contained in the ball $B_{r_0} = \{x \in \mathbb{R}^N \mid |x| < r_0\}$ with some $r_0 > 0$. We never impose any geometric condition on the domain Ω.

We investigate L^p decay estimates with $p \geqslant 1$ of the solution to the initial–boundary value problem for the linear dissipative wave equation:

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$$\begin{cases} (\Box + \partial_t)u = 0, \quad u = u(x,t), \quad \text{in } \Omega \times (0,\infty), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \quad \text{and} \quad u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Box + \partial_t = \partial_t^2 + \partial_t - \Delta$ is the dissipative wave operator, $\partial_t = \partial/\partial t$, and $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial_{x_j}^2$ is the Laplacian.

In order to derive the decay estimates to exterior domain problems, in addition to the decay estimates to the Cauchy problem in the whole space, we need to prepare appropriate local energy decay estimates of higher order. On the one side, the local energy decay estimate of first order for Eq. (1.1) is given by Dan and Shibata [2] (see Proposition 2.1 in Section 2), and then, by induction argument based on Dan and Shibata's work [2], we can obtain the following.

Theorem 1.1 *(Higher local energy decay). Let $N \geqslant 2$ and $m \geqslant 1$ be integers. Let $r > r_0$. Suppose that the initial data $u_0 \in H^m(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H^{m-1}(\Omega)$ satisfy the compatibility condition of order $m-1$ and*

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega_r,$$

where $\Omega_r = \Omega \cap B_r$ with the ball $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$. Then, the solution $u(t)$ of Eq. (1.1) satisfies that for $r_0 < r_1 < r$ and for $0 \leqslant k \leqslant m$,

$$\left\|\partial_t^k u(t)\right\|_{H^{m-k}(\Omega_{r_1})} \leqslant C_{r_1,m}(1+t)^{-N/2}\left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)}\right) \tag{1.2}$$

for $t \geqslant 0$, where $C_{r_1,m}$ is a constant depending on r_1 and m.

Remark. "Compatibility condition" is given as follows. Inductively, we define u_k for $k \geqslant 2$ by $u_k = \Delta u_{k-2} - u_{k-1}$. Then, we say the initial data $u_0 \in H^m(\Omega)$ and $u_1 \in H^{m-1}(\Omega)$ satisfy the compatibility condition of order $m-1$ if $u_i \in H_0^1(\Omega)$ and $u_{i+1} \in L^2(\Omega)$ for $0 \leqslant i \leqslant m-1$.

On the other side, the L^p decay estimates with $p \geqslant 1$ to the Cauchy problem in the whole space have been given by the previous papers [18] for even dimensions and [19] for odd dimensions, together with the representation formulas of the solutions for each dimensions (see Proposition 2.2 in Section 2, cf. Marcati and Nishihara [6], Nishihara [13], and Ono [14,15] for lower dimensions), and the L^2 decay and higher energy decay estimates to the Cauchy problem have been shown by Matsumura [7] (see Proposition 2.1 in Section 2).

Thus, by the cut-off technique, we have the following L^p decay estimates with $1 \leqslant p < 2$ for the exterior problem (1.1).

Theorem 1.2. *Let $N \geqslant 3$ be an integer, and let $n = [N/2]$ be the integer part of $N/2$. Suppose that the initial data $u_0 \in H^n(\Omega) \cap H_0^1(\Omega) \cap W^{n,1}(\Omega)$ and $u_1 \in H^{n-1}(\Omega) \cap W^{n-1,1}(\Omega)$ satisfy the compatibility condition of order $n-1$. Then, the solution $u(t)$ of Eq. (1.1) satisfies that for $1 \leqslant p < 2$,*

$$\left\|u(t)\right\|_{L^p(\Omega)} \leqslant C d_n (1+t)^{-(N/2)(1-1/p)} \tag{1.3}$$

for $t \geqslant 0$, where d_n is given by

$$d_n = \|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)} + \|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)}. \tag{1.4}$$

Remark. For the two-dimensional case $N = 2$, we have treated the L^p estimates of the solution $u(t)$ in [16], in particular, when $p = 1$, it holds that

$$\|u(t)\|_{L^1(\Omega)} \leqslant C(1+t)^{\delta}\big(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{W^{1,1}(\Omega)} + \|u_1\|_{L^1(\Omega)}\big)$$

for $t \geqslant 0$ with any small $\delta > 0$.

For the total energy

$$E(t;\Omega) \equiv (1/2)\|\partial_t u(t)\|^2_{L^2(\Omega)} + (1/2)\|\nabla u(t)\|^2_{L^2(\Omega)}$$

associated with Eq. (1.1) with the initial data (u_0, u_1) belonging to $H^1_0(\Omega) \times L^2(\Omega)$, it is well known that the standard energy method yields the energy decay estimate

$$E(t;\Omega)^{1/2} \leqslant C(1+t)^{-1/2}\big(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}\big)$$

for $t \geqslant 0$. Recently, Nakao [11] has derived energy decay estimates of higher order for quasilinear wave equations with a localized dissipation, e.g., for $1 \leqslant k \leqslant m$ (when $m > [N/2] + 1$),

$$\|\partial_t^k u(t)\|_{H^{m-k}(\Omega)} + \|\partial_t^{k-1}\nabla u(t)\|_{H^{m-k}(\Omega)} \leqslant C(1+t)^{-k/2}$$

with the initial data belonging to higher energy space $H^m(\Omega) \cap H^1_0(\Omega) \times H^{m-1}(\Omega)$ and satisfying the compatibility condition of order $m-1$.

However, if the initial data belong to L^1 spaces, we can improve the decay rate of the energy decay estimates of higher order for Eq. (1.1). Indeed, in previous paper [16], we have shown that the total energy

$$E(t;\Omega)^{1/2} \leqslant C(1+t)^{-1/2-N/4}\big(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{L^1(\Omega)} + \|u_1\|_{L^1(\Omega)}\big)$$

for $t \geqslant 0$, and then, by induction argument based on the above improved total energy decay estimate, we can obtain the following.

Theorem 1.3. *Let $N \geqslant 3$ and $m \geqslant 1$ be integers. Suppose that $u_0 \in H^m(\Omega) \cap H^1_0(\Omega) \cap L^1(\Omega)$ and $u_1 \in H^{m-1}(\Omega) \cap L^1(\Omega)$ satisfy the compatibility condition of order $m-1$. Then, the solution $u(t)$ of Eq.* (1.1) *satisfies that for $1 \leqslant k \leqslant m$,*

$$\|\partial_t^k u(t)\|_{H^{m-k}(\Omega)} + \|\partial_t^{k-1}\nabla u(t)\|_{H^{m-k}(\Omega)} \leqslant C d_{m,1}(1+t)^{-k/2-N/4} \tag{1.5}$$

and

$$\|u(t)\|_{H^m(\Omega)} \leqslant C d_{m,1}(1+t)^{-N/4} \tag{1.6}$$

for $t \geqslant 0$, where $d_{m,1}$ is given by

$$d_{m,1} = \|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} + \|u_0\|_{L^1(\Omega)} + \|u_1\|_{L^1(\Omega)}. \tag{1.7}$$

Remark. The decay estimates of higher order $m \geqslant 2$ for the solution of Eq. (1.1) in Theorem 1.3 correct the mistakes in previous paper [16].

This paper is organized as follows. In Section 2, we state decay estimates of the solution to the Cauchy problem in the whole space of the linear dissipative wave equation. In Section 3, we consider the local energy estimates of higher order to the initial–boundary value problem (1.1). In Section 4, we consider the L^1 estimates of the solution of Eq. (1.1) in dimension $N \geqslant 3$. In Section 5, we give the proof of Theorem 1.3.

We use only familiar functional spaces ($L^p(\Omega)$, $W^{m,p}(\Omega)$, $H^m(\Omega)$, etc.) and omit the definitions. Given a multiindex $\beta = (\beta_1, \dots, \beta_m)$ of order $|\beta| = \beta_1 + \cdots + \beta_m$, we define $D_x^\beta u = \partial_{x_1}^{\beta_1} \cdots \partial_{x_m}^{\beta_m} u$. If ℓ is a nonnegative integer, $\nabla^\ell u = \{D_x^\beta u \mid |\beta| = \ell\}$ stands for the set of all partial derivatives of order ℓ, and $|\nabla^\ell u| = (\sum_{|\beta|=\ell} |D_x^\beta u|^2)^{1/2}$. We note that $\|u\|_{W^{m,p}(\Omega)} = \sum_{\ell=0}^m \|\nabla^\ell u\|_{L^p(\Omega)}$ where $\|\cdot\|_{L^p(\Omega)}$ denotes the usual L^p-norm, and $H_0^1(\Omega)$ is a completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. We set $L^p(\Omega) = W^{0,p}(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. Positive constants will be denoted by C and will change from line to line.

2. Preliminaries

First we consider the Cauchy problem in the whole space (i.e., $\Omega = \mathbb{R}^N$ for $N \geqslant 1$) of the linear dissipative wave equation:

$$\begin{cases} (\Box + \partial_t)v = 0, \quad v = v(x,t), \quad \text{in } \mathbb{R}^N \times (0,\infty), \\ (v, \partial_t v)|_{t=0} = (v_0, v_1). \end{cases} \tag{2.1}$$

The following L^2 decay and higher energy estimates of the solution $v(t)$ of Eq. (2.1) are well known, and were shown by Matsumura [7] (also see [5,16]).

Proposition 2.1. *Let $m \geqslant 0$ be zero or an integer. Suppose that initial data $v_0 \in H^m(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $v_1 \in H^{m-1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Then, the solution $v(t)$ of Eq.* (2.1) *satisfies that for $0 \leqslant k + b \leqslant m$,*

$$\left\|\partial_t^k \nabla^b v(t)\right\|_{L^2(\mathbb{R}^N)} \leqslant C\bar{d}_{m,1}(1+t)^{-k-b/2-N/4} \tag{2.2}$$

for $t \geqslant 0$, where $\bar{d}_{m,1}$ is given by

$$\bar{d}_{m,1} = \|v_0\|_{H^m(\mathbb{R}^N)} + \|v_1\|_{H^{m-1}(\mathbb{R}^N)} + \|v_0\|_{L^1(\mathbb{R}^N)} + \|v_1\|_{L^1(\mathbb{R}^N)}. \tag{2.3}$$

Recently, in [18,19], respectively for even and odd dimensions, we have derived the following L^1 estimates for the solution $v(t)$ of Eq. (2.1), where we can see the same results with lower dimensions in [14,15] for $N \leqslant 3$ and [17] for $N \leqslant 5$ (also see Marcati and Nishihara [6], Nishihara [13] for $N = 1, 3$, cf. Milani and Han [8] for $t \gg 1$). In the proof we use the representation formulas of the solution $v(t)$ (cf. Courant and Hilbert [1]).

Proposition 2.2. *Let $n = [N/2]$ be the integer part of $N/2$. Suppose that the initial data $v_0 \in W^{n,1}(\mathbb{R}^N)$ and $v_1 \in W^{n-1,1}(\mathbb{R}^N)$ ($v_0 \in L^1(\mathbb{R}^N)$ and $v_1 \in L^1(\mathbb{R}^N)$ if $N = 1$). Then, the solution $v(t)$ of Eq.* (2.1) *satisfies that*

$$\left\|v(t)\right\|_{L^1(\mathbb{R}^N)} \leqslant C\big(\|v_0\|_{W^{n,1}(\mathbb{R}^N)} + \|v_1\|_{W^{n-1,1}(\mathbb{R}^N)}\big) \tag{2.4}$$

($\|v(t)\|_{L^1(\mathbb{R}^N)} \leqslant C(\|v_0\|_{L^1(\mathbb{R}^N)} + \|v_1\|_{L^1(\mathbb{R}^N)})$ *if $N = 1$) for $t \geqslant 0$.*

3. Local energy decay

In this section we shall prove Theorem 1.1. First we state the local energy decay estimate of the first order for Eq. (1.1), which was proved by Dan and Shibata [2] (cf. Shibata and Tsutsumi [20] for $N \geqslant 3$).

Proposition 3.1 *(Local energy decay). Let $N \geqslant 2$ be an integer and let $r > r_0$. Suppose that the initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ and*

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega_r,$$

where $\Omega_r = \Omega \cap B_r$. Then, the solution $u(t)$ of Eq. (1.1) *satisfies that*

$$\left\|u(t)\right\|_{H^1(\Omega_r)} + \left\|\partial_t u(t)\right\|_{L^2(\Omega_r)} \leqslant C_r(1+t)^{-N/2}\left(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}\right) \tag{3.1}$$

for $t \geqslant 0$, where C_r is a constant depending on r.

By induction technique together with Proposition 3.1, we can obtain local energy decay estimates of higher order.

Proposition 3.2. *Let $m \geqslant 1$ be an integer. In addition to the assumptions of Proposition* 3.1*, suppose that the initial data $u_0 \in H^m(\Omega)$ and $u_1 \in H^{m-1}(\Omega)$ satisfy the compatibility condition of order $m-1$* (*that is, the assumptions of Theorem* 1.1 *are fulfilled*)*. Then, the solution $u(t)$ of Eq.* (1.1) *satisfies that for $1 \leqslant k \leqslant m$,*

$$\begin{aligned}&\left\|\partial_t^{k-1} u(t)\right\|_{H^1(\Omega_r)} + \left\|\partial_t^k u(t)\right\|_{L^2(\Omega_r)} \\ &\quad \leqslant C_{r,m}(1+t)^{-N/2}\left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)}\right)\end{aligned} \tag{3.2}$$

for $t \geqslant 0$, where $C_{r,m}$ is a constant depending on r and m.

Proof. Put $w = \partial_t^{k-1} u$, then the function w satisfies $(\Box + \partial_t)w = 0$ with $w|_{\partial\Omega} = 0$. Thus, by Proposition 3.1 above together with the compatibility condition of order $k-1$, we have that

$$\left\|w(t)\right\|_{H^1(\Omega_r)} + \left\|\partial_t w(t)\right\|_{L^2(\Omega_r)} \leqslant C_r(1+t)^{-N/2}\left(\left\|w(0)\right\|_{H^1(\Omega)} + \left\|\partial_t w(0)\right\|_{L^2(\Omega)}\right)$$

or

$$\left\|\partial_t^{k-1} u(t)\right\|_{H^1(\Omega_r)} + \left\|\partial_t^k u(t)\right\|_{L^2(\Omega_r)} \leqslant C_{r,m}(1+t)^{-N/2}\left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)}\right)$$

for $t \geqslant 0$. $\Box$

Take any r_1 such that $r_0 < r_1 < r$, and take any r_k $(k = 2, 3, \ldots, m-1)$ such that

$$r_0 < r_1 < r_2 < \cdots < r_{m-1} < r_m = r.$$

And, we choose a smooth function $\eta_k(x)$ $(k = 1, 2, \ldots, m-1)$ such that $0 \leqslant \eta_k(x) \leqslant 1$ and

$$\eta_k(x) = \begin{cases} 1 & \text{if } |x| \leqslant r_k, \\ 0 & \text{if } |x| \geqslant r_{k+1}. \end{cases}$$

Put $w_k = \partial_t^k u$ for $k \geqslant 0$, then the function $(\eta_{k+1} w_k)$ satisfies that

$$\begin{cases} (\Box + \partial_t)(\eta_{k+1} w_k) = -f_{k+1} & \text{in } \Omega_{r_{k+2}} \times (0, \infty), \\ (\eta_{k+1} w_k)|_{\partial\Omega_{r_{k+2}}} = 0, \end{cases} \tag{3.3}$$

where $f_{k+1} = 2\nabla\eta_{k+1} \cdot \nabla w_k + \Delta\eta_k \cdot w_k$ with $\operatorname{supp} f_{k+1} \subset \{x \in \mathbb{R}^N \mid r_{k+1} \leqslant |x| \leqslant r_{k+2}\}$.

Proposition 3.3. *Suppose that the assumptions of Theorem* 3.2 (*or Theorem* 1.1) *are fulfilled. Let* $1 \leqslant \ell \leqslant m$. *If the solution* $v(t)$ *of Eq.* (1.1) *satisfies that for* $0 \leqslant k \leqslant \ell - 1$,

$$\left\| \partial_t^{m-k} u(t) \right\|_{H^k(\Omega_{r_{m-k+1}})} \leqslant C_{r_1,m} (1+t)^{-N/2} \left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} \right), \tag{3.4}$$

then it holds that

$$\left\| \partial_t^{m-\ell} u(t) \right\|_{H^\ell(\Omega_{r_{m-\ell+1}})} \leqslant C_{r_1,m} (1+t)^{-N/2} \left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} \right) \tag{3.5}$$

for $t \geqslant 0$.

Proof. Since it follows from (3.3) that

$$\Delta\left(\eta_{m-\ell+1} \partial_t^{m-\ell} u\right) = \eta_{m-\ell+1} \left(\partial_t^{m-\ell+2} u + \partial_t^{m-\ell+1} u + f_{m-\ell+1} \right) \equiv F_{m+\ell+1}$$

with $(\eta_{m-\ell+1} \partial_t^{m-\ell} u)|_{\partial \Omega_{r_{m-\ell+2}}} = 0$, we observe from Lemma 3.4 below that

$$\begin{aligned}
\left\| \partial_t^{m-\ell} u(t) \right\|_{H^\ell(\Omega_{r_{m-\ell+1}})} &\leqslant \left\| \eta_{m-\ell+1} \partial_t^{m-\ell} u(t) \right\|_{H^\ell(\Omega_{r_{m-\ell+2}})} \\
&\leqslant \left\| F_{m-\ell+1}(t) \right\|_{H^{\ell-1}(\Omega_{r_{m-\ell+2}})} + \left\| \eta_{m-\ell+1} \partial_t^{m-\ell} u(t) \right\|_{L^2(\Omega_{r_{m-\ell+2}})} \\
&\leqslant C_{r_1,m} (1+t)^{-N/2} \left(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} \right),
\end{aligned}$$

where we used (3.4) and (3.2) in the last inequality. □

Proof of Theorem 1.1. By induction technique, Theorem 1.1 follows from Propositions 3.2 and 3.3. □

We state the regularity theorem in exterior domains (see, e.g., [3,4,20]).

Lemma 3.4. *Let* $m \geqslant 2$ *be an integer and* w *belong to* $H^m(\Omega) \cap H_0^1(\Omega)$. *Then, it holds that*

$$\|\nabla w\|_{H^{m-1}(\Omega)} \leqslant C \|\Delta w\|_{H^{m-2}(\Omega)} + C \|\nabla w\|_{L^2(\Omega)} \tag{3.6}$$

and

$$\|w\|_{H^m(\Omega)} \leqslant C \|\Delta w\|_{H^{m-2}(\Omega)} + C \|w\|_{L^2(\Omega)}. \tag{3.7}$$

Proof. We will give the proof of the regularity theorem in exterior domains, by using the regularity theorem in interior domains (see [3,4]).

Let $r > r_0$. Choose a smooth function $\eta(x)$ such that $0 \leqslant \eta(x) \leqslant 1$ and

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \leqslant r, \\ 0 & \text{if } |x| \geqslant r+1. \end{cases}$$

Then, we observe from the regularity theorem in interior domains that for $w \in H^m(\Omega) \cap H_0^1(\Omega)$ when $m \geqslant 2$,

$$\left\| \nabla^2 w \right\|_{H^{m-2}(\Omega_r)} \leqslant \left\| \nabla^2 (\eta w) \right\|_{H^{m-2}(\Omega_{r+1})} \leqslant C \left\| \Delta(\eta w) \right\|_{H^{m-2}(\Omega_{r+1})},$$

where $\Omega_r = \Omega \cap B_r$ with the ball $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$, and by the Poincaré inequality,

$$\begin{aligned}\|\nabla^2 w\|_{H^{m-2}(\Omega_r)} &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)} + C\|w\|_{H^{m-2}(\Omega_{r+1})}\\ &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)}.\end{aligned}$$

Next, we estimate $|\nabla^2 w|$ outside of the domain Ω_r. Choose a smooth function $\chi_0(x)$ such that $0 \leqslant \chi_0(x) \leqslant 1$ and

$$\chi_0(x) = \begin{cases} 0 & \text{if } |x| \leqslant r_0,\\ 1 & \text{if } |x| \geqslant r.\end{cases}$$

Then, we observe that

$$\|\nabla^2 w\|_{H^{m-2}(\Omega_r^c)} \leqslant \|\nabla^2(\chi_0\bar{w})\|_{H^{m-2}(\mathbb{R}^N)} \leqslant C\|\Delta(\chi_0\bar{w})\|_{H^{m-2}(\mathbb{R}^N)},$$

where $\Omega_r^c = \mathbb{R}^N \setminus \Omega_r$ and $\bar{w}$ is a function on $\mathbb{R}^N$ with $\bar{w} = w$ on Ω, and by the Poincaré inequality,

$$\begin{aligned}\|\nabla^2 w\|_{H^{m-2}(\Omega_r^c)} &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)} + C\|w\|_{H^{m-2}(\Omega_r)}\\ &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)}.\end{aligned}$$

Summing up the above estimates, we have that

$$\|\nabla^2 w\|_{H^{m-2}(\Omega)} \leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)} \tag{3.8}$$

or

$$\|\nabla w\|_{H^{m-1}(\Omega)} \leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)}. \tag{3.9}$$

Now, we will establish (3.6) by induction on m, the case $m = 2$ being the estimate (3.9) above. Assume now assertion (3.6) is valid for $m - 1$ when $m \geqslant 3$, that is,

$$\|\nabla w\|_{H^{m-2}(\Omega)} \leqslant C\|\Delta w\|_{H^{m-3}(\Omega)} + C\|\nabla w\|_{L^2(\Omega)}.$$

By the estimate (3.8) and the induction hypotheses, we have that

$$\begin{aligned}\|\nabla w\|_{H^{m-1}(\Omega)} &\leqslant \|\nabla^2 w\|_{H^{m-2}(\Omega)} + \|\nabla w\|_{H^{m-2}(\Omega)}\\ &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{H^{m-2}(\Omega)}\\ &\leqslant C\|\Delta w\|_{H^{m-2}(\Omega)} + C\|\nabla w\|_{L^2(\Omega)}\end{aligned}$$

and hence, we conclude the desired estimate (3.6) for any $m \geqslant 2$.

Moreover, using the inequality

$$\|\nabla w\|_{L^2(\Omega)}^2 \leqslant \frac{1}{2}\|\Delta w\|_{L^2(\Omega)}^2 + \frac{1}{2}\|w\|_{L^2(\Omega)}^2,$$

we obtain the desired estimate (3.7) for any $m \geqslant 2$. □

4. L^1 estimates

In this section we shall give the proof of Theorem 1.2. In order to prove Theorem 1.2, we use the so-called cut-off method as in Shibata and Tsutsumi [20] and Nakao [10], and we use the following L^2 decay and energy decay estimates which were shown by previous paper [16].

Proposition 4.1. *Let $N \geqslant 3$ be an integer. Suppose that the initial data $u_0 \in H_0^1(\Omega) \cap L^1(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^1(\Omega)$. Then, the solution $u(t)$ of Eq.* (1.1) *satisfies that*

$$\|u(t)\|_{L^2(\Omega)} \leqslant C d_{1,1}(1+t)^{-N/4} \tag{4.1}$$

and

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_{1,1}(1+t)^{-1/2-N/4} \tag{4.2}$$

for $t \geqslant 0$, where $d_{1,1}$ is given by

$$d_{1,1} = \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{L^1(\Omega)} + \|u_1\|_{L^1(\Omega)}.$$

As a cut-off function in $\mathbb{R}^N$, we choose a smooth function $\chi(x)$ such that $0 \leqslant \chi(x) \leqslant 1$ and

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \leqslant r, \\ 1 & \text{if } |x| \geqslant r+1, \end{cases}$$

and we consider the initial–boundary value problem

$$\begin{cases} (\Box + \partial_t)u_\chi = 0 & \text{in } \Omega \times (0,\infty), \\ (u_\chi, \partial_t u_\chi)|_{t=0} = (\chi u_0, \chi u_1) & \text{and} \quad u_\chi|_{\partial\Omega} = 0. \end{cases} \tag{4.3}$$

Then, from Proposition 4.1 we see that the solution $u_\chi(t)$ of Eq. (4.3) satisfies the H^1 estimate

$$\|u_\chi(t)\|_{H^1(\Omega)} \leqslant C d_{1,1}(1+t)^{-N/4} \quad \text{for } t \geqslant 0. \tag{4.4}$$

Moreover, we have the following the L^1 estimate of the solution $u_\chi(t)$ of Eq. (4.3).

Proposition 4.2. *In addition to the assumptions of Proposition* 4.1*, suppose that the initial data $u_0 \in H^n(\Omega) \cap W^{n,1}(\Omega)$ and $u_1 \in H^{n-1}(\Omega) \cap W^{n-1,1}(\Omega)$ satisfy the compatibility condition of order $n-1$ (that is, the assumptions of Theorem* 1.2 *are fulfilled). Then, the solution $u_\chi(t)$ of Eq.* (4.3) *satisfies that*

$$\|u_\chi(t)\|_{L^1(\Omega)} \leqslant C\big(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)} + \|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)}\big) \tag{4.5}$$

for $t \geqslant 0$.

Proof of Proposition 4.2. In order to derive the L^1 estimate of the function $u_\chi(t)$, we introduce the solution $v(t)$ of the Cauchy problem

$$\begin{cases} (\Box + \partial_t)v = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ (v, \partial_t v)|_{t=0} = (\bar{u}_0, \bar{u}_1), \end{cases}$$

where $\bar{f}$ is a function in $\mathbb{R}^N$ such that

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases} \tag{4.6}$$

Then, from Propositions 2.1 and 2.2, we have that for $1 \leqslant m \leqslant n$,

$$\|\nabla v(t)\|_{H^{m-1}(\mathbb{R}^N)} \leqslant C d_{m,1}(1+t)^{-1/2-N/4} \tag{4.7}$$

with $d_{m,1} = \|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)} + \|u_0\|_{L^1(\Omega)} + \|u_1\|_{L^1(\Omega)}$ and

$$\|v(t)\|_{L^1(\mathbb{R}^N)} \leqslant C\big(\|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)}\big) \tag{4.8}$$

for $t \geqslant 0$. Moreover, we observe that the function $\chi v(t)$ satisfies

$$\begin{cases} (\Box + \partial_t)(\chi v) = g & \text{in } \Omega \times (0,\infty), \\ (\chi v, \partial_t \chi v)|_{t=0} = (\chi u_0, \chi u_1) \quad \text{and} \quad (\chi v)|_{\partial\Omega} = 0, \end{cases}$$

where $g = -2\nabla\chi \cdot \nabla v - \Delta\chi \cdot v$ with $\operatorname{supp} g \subset \{x \in \mathbb{R}^N \mid r \leqslant |x| \leqslant r+1\}$, and the function $w(t) = u_\chi(t) - \chi v(t)$ satisfies

$$\begin{cases} (\Box + \partial_t) w = -g & \text{in } \Omega \times (0,\infty), \\ (w, \partial_t w)|_{t=0} = (0,0) \quad \text{and} \quad w|_{\partial\Omega} = 0. \end{cases}$$

Here, we denote the solution to the initial–boundary value problem of Eq. (1.1) by $S(t; \{u_0, u_1\})$, and then, by the Duhamel principle (e.g., [3]), we see that

$$w(t) = \int_0^t S\big(t-s; \{0, -g(s)\}\big)\, ds.$$

Applying Theorem 1.1 to the function $w(t)$ in the domain $\Omega_{r+3} = \Omega \cap B_{r+3}$, we have that for $1 \leqslant m \leqslant n$,

$$\begin{aligned} \|w(t)\|_{H^m(\Omega_{r+3})} &\leqslant \int_0^t \big\|S\big(t-s; \{0, -g(s)\}\big)\big\|_{H^m(\Omega_{r+3})}\, ds \\ &\leqslant \int_0^t (1+t-s)^{-N/2} \|g(s)\|_{H^{m-1}(\Omega)}\, ds. \end{aligned}$$

Here, it follows from the Hölder inequality and (4.7) that

$$\begin{aligned} \|g(t)\|_{H^{m-1}(\Omega)} &\leqslant C\|\nabla v(t)\|_{H^{m-1}(\Omega_{r+1})} + C\|v(t)\|_{W^{m-1,2N/(N-2)}(\Omega_{r+1})} \\ &\leqslant C\|\nabla v(t)\|_{H^{m-1}(\mathbb{R}^N)} \\ &\leqslant C d_{m,1}(1+t)^{-1/2-N/4}. \end{aligned}$$

Thus, we obtain that for $1 \leqslant m \leqslant n$,

$$\|w(t)\|_{H^m(\Omega_{r+3})} \leqslant C d_{m,1}(1+t)^{-1/2-N/4} \tag{4.9}$$

and

$$\|w(t)\|_{L^1(\Omega_{r+3})} \leqslant C\|w(t)\|_{H^1(\Omega_{r+3})} \leqslant C d_{1,1}(1+t)^{-1/2-N/4}. \tag{4.10}$$

Next, we shall estimate the L^1 norm of the function $w(t)$ outside of the domain Ω_{r+3}. As a cut-off function in $\mathbb{R}^N$, we choose a smooth function $\chi_2(x)$ such that $0 \leqslant \chi_2(x) \leqslant 1$ and

$$\chi_2(x) = \begin{cases} 0 & \text{if } |x| \leqslant r+2, \\ 1 & \text{if } |x| \geqslant r+3. \end{cases}$$

Then, since the function $\bar{w} = \bar{u}_\chi - \chi v$ satisfies

$$\begin{cases} (\Box + \partial_t)\bar{w} = -g & \text{in } \mathbb{R}^N \times (0,\infty), \\ (\bar{w}, \partial_t \bar{w})|_{t=0} = (0,0), \end{cases}$$

we observe that the function $\chi_2 \bar{w}(t)$ satisfies

$$\begin{cases} (\Box + \partial_t)(\chi_2 \bar{w}) = h & \text{in } \mathbb{R}^N \times (0,\infty), \\ (\chi_2 \bar{w}, \partial_t \chi_2 \bar{w})|_{t=0} = (0,0), \end{cases}$$

where $h = -\chi_2 g - 2\nabla\chi_2 \cdot \nabla\bar{w} - \Delta\chi_2 \cdot \bar{w} = -2\nabla\chi_2 \cdot \nabla\bar{w} - \Delta\chi_2 \cdot \bar{w}$ with $\operatorname{supp} h \subset \{x \in \mathbb{R}^N \mid r+2 \leqslant |x| \leqslant r+3\}$.

Here, we denote the solution to the Cauchy problem of Eq. (2.1) by $\widetilde{S}(t; \{v_0, v_1\})$, and then, by Duhamel principle, we see that

$$\chi_2 \bar{w}(t) = \int_0^t \widetilde{S}\bigl(t-s; \{0, h(s)\}\bigr)\, ds.$$

Applying Proposition 2.2 to the function $\chi_2 \bar{w}(t)$, we have that

$$\begin{aligned} \|w(t)\|_{L^1(\Omega_{r+3}^c)} \leqslant \|\chi_2 \bar{w}(t)\|_{L^1(\mathbb{R}^N)} &\leqslant \int_0^t \bigl\|\widetilde{S}\bigl(t-s; \{0, h(s)\}\bigr)\bigr\|_{L^1(\mathbb{R}^N)}\, ds \\ &\leqslant C \int_0^t \|h(s)\|_{W^{n-1,1}(\mathbb{R}^N)}\, ds. \end{aligned}$$

Here, it follows from (4.9) that

$$\|h(t)\|_{W^{n-1,1}(\mathbb{R}^N)} \leqslant C\|h(t)\|_{H^{n-1}(\Omega_{r+3})} \leqslant C\|w(t)\|_{H^n(\Omega_{r+3})} \leqslant C d_{n,1}(1+t)^{-1/2-N/4}.$$

Thus, we obtain that

$$\|w(t)\|_{L^1(\Omega_{r+3}^c)} \leqslant C d_{n,1} \quad \text{for } t \geqslant 0. \tag{4.11}$$

Therefore, from (4.8), (4.10) and (4.11) we have the L^1 estimate of the function $u_\chi = \chi v + w$, that is,

$$\begin{aligned} \|u_\chi(t)\|_{L^1(\Omega)} &\leqslant C\|v(t)\|_{L^1(\Omega)} + \|w(t)\|_{L^1(\Omega_{r+3})} + \|w(t)\|_{L^1(\Omega_{r+3}^c)} \\ &\leqslant C\bigl(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)} + \|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)}\bigr) \end{aligned}$$

for $t \geqslant 0$. $\Box$

Proof of Theorem 1.2. Thanks to the L^2 estimate (4.1) of the solution $u(t)$ for Eq. (1.1), it is enough to derive the L^1 estimate of the solution $u(t)$.

On the one side, since we have the L^1 estimate (4.5) of the solution $u_\chi(t)$ for Eq. (4.3), we shall derive the L^1 estimate of the function $U = u - u_\chi$.

It is easy to see that the function $U(t)$ satisfies

$$\begin{cases} (\Box + \partial_t)U = 0 & \text{in } \Omega \times (0,\infty), \\ (U, \partial_t U)|_{t=0} = ((1-\chi)u_0, (1-\chi)u_1) \quad \text{and} \quad U|_{\partial\Omega} = 0 \end{cases}$$

and then, by Theorem 1.1 again, we observe that for $1 \leqslant m \leqslant n$,

$$\begin{aligned}\|U(t)\|_{L^1(\Omega_{r+3})} &\leqslant C\|U(t)\|_{H^m(\Omega_{r+3})} \\ &\leqslant C(1+t)^{-N/2}\big(\|u_0\|_{H^m(\Omega)} + \|u_1\|_{H^{m-1}(\Omega)}\big),\end{aligned} \tag{4.12}$$

and moreover, we see that the function $\chi_2\bar{U}(t)$ satisfies

$$\begin{cases}(\Box + \partial_t)(\chi_2\bar{U}) = f & \text{in } \mathbb{R}^N \times (0,\infty),\\ (\chi_2\bar{U}, \partial_t(\chi_2\bar{U}))|_{t=0} = (0,0),\end{cases}$$

where $f = -2\nabla\chi_2 \cdot \nabla\bar{U} - \Delta\chi_2 \cdot \bar{U}$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^N \mid r+2 \leqslant |x| \leqslant r+3\}$. Since it follows that

$$\chi_2\bar{U}(t) = \int_0^t \widetilde{S}\big(t-s; \{0, f(s)\}\big)\, ds,$$

applying Proposition 2.2 to the function $\chi_2\bar{U}(t)$, we have that

$$\begin{aligned}\|U(t)\|_{L^1(\Omega_{r+3}^c)} \leqslant \|\chi_2\bar{U}(t)\|_{L^1(\mathbb{R}^N)} &\leqslant \int_0^t \big\|\widetilde{S}\big(t-s; \{0, f(s)\}\big)\big\|_{L^1(\mathbb{R}^N)}\, ds \\ &\leqslant C\int_0^t \|f(s)\|_{W^{n-1,1}(\mathbb{R}^N)}\, ds.\end{aligned}$$

Here, we observe from (4.12) that

$$\begin{aligned}\|f(t)\|_{W^{n-1,1}(\mathbb{R}^N)} &\leqslant C\|f(t)\|_{H^{n-1}(\Omega_{r+3})} \leqslant C\|U(t)\|_{H^n(\Omega_{r+3})} \\ &\leqslant C(1+t)^{-N/2}\big(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)}\big).\end{aligned}$$

Thus, we obtain that

$$\|U(t)\|_{L^1(\Omega_{r+3}^c)} \leqslant C\big(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)}\big) \tag{4.13}$$

for $t \geqslant 0$.

Therefore, from (4.5), (4.12) and (4.13) we have the L^1 estimate of the function $u = u_\chi + U$, that is,

$$\begin{aligned}\|u(t)\|_{L^1(\Omega)} &\leqslant \|u_\chi(t)\|_{L^1(\Omega)} + \|U(t)\|_{L^1(\Omega_{r+3})} + \|U(t)\|_{L^1(\Omega_{r+3}^c)} \\ &\leqslant C\big(\|u_0\|_{H^n(\Omega)} + \|u_1\|_{H^{n-1}(\Omega)} + \|u_0\|_{W^{n,1}(\Omega)} + \|u_1\|_{W^{n-1,1}(\Omega)}\big)\end{aligned}$$

for $t \geqslant 0$, and hence, due to this estimate and the L^2 estimate (4.1), we obtain the desired estimate (1.3). □

5. Higher energy decay estimates

In this section we shall give the proof of Theorem 1.3. The following lemma connected with the energy decay is very useful for the proof of Theorem 1.3, and it follows from the Nakao inequality in [9] (e.g., see [5,12,16] for the proof).

Lemma 5.1. *Let u be a solution of Eq.* (1.1). *Suppose that*

$$\|u(t)\|_{L^2(\Omega)} \leqslant A_0(1+t)^{-a} \quad \text{for } t \geqslant 0,$$

with some constant A_0 and $a \geqslant 0$. Then, it holds that

$$\|\partial_t u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_1'(1+t)^{-1/2-a}$$

for $t \geqslant 0$, where $d_1' = A_0 + \|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}$.

Proposition 5.2. *Let $m \geqslant 1$ be an integer. Suppose that the assumptions of Theorem* 1.3 *are fulfilled. Then, the solution $u(t)$ of Eq.* (1.1) *satisfies that for $1 \leqslant k \leqslant m$,*

$$\|\partial_t^k u(t)\|_{L^2(\Omega)} + \|\partial_t^{k-1}\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_{m,1}(1+t)^{-k/2-N/4} \tag{5.1}$$

for $t \geqslant 0$, where $d_{m,1}$ is given by (1.7).

Proof. We shall establish (5.1) by induction on k, the case $k = 1$ being Proposition 4.1 in previous section.

Let ℓ be an integer such that $2 \leqslant \ell \leqslant m$. We now assume that for $1 \leqslant k \leqslant \ell - 1$,

$$\|\partial_t^k u(t)\|_{L^2(\Omega)} + \|\partial_t^{k-1}\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_{m,1}(1+t)^{-k/2-N/4}.$$

Put $w = \partial_t^{\ell-1} u$ ($k = \ell - 1$), then the function w satisfies $(\Box + \partial_t)w = 0$ with $w|_{\partial\Omega} = 0$, and

$$\|w(t)\|_{L^2(\Omega)} \leqslant C d_{m,1}(1+t)^{-(\ell-1)/2-N/4}.$$

Thus, applying Lemma 5.1 to the function w, we obtain that

$$\begin{aligned}\|\partial_t^\ell u(t)\|_{L^2(\Omega)} + \|\partial_t^{\ell-1}\nabla u(t)\|_{L^2(\Omega)} &= \|\partial_t w(t)\|_{L^2(\Omega)} + \|\nabla w(t)\|_{L^2(\Omega)} \\ &\leqslant C d_{m,1}(1+t)^{-\ell/2-N/4},\end{aligned}$$

and hence, the desired estimate (5.1) is valid for $1 \leqslant k \leqslant m$. $\Box$

Proposition 5.3. *Let $m \geqslant 1$ be an integer. Suppose that the assumptions of Theorem* 1.3 *are fulfilled. Then, the solution $u(t)$ of Eq.* (1.1) *satisfies that for $0 \leqslant j \leqslant m - 1$,*

$$\|\partial_t^{m-j-1}\nabla u(t)\|_{H^j(\Omega)} \leqslant C d_{m,1}(1+t)^{-(m-j)/2-N/4} \tag{5.2}$$

for $t \geqslant 0$, where $d_{m,1}$ is given by (1.7).

Proof. We shall establish (5.2) by induction on j. From Proposition 5.2, it holds that for $1 \leqslant k \leqslant m$,

$$\|\partial_t^k u(t)\|_{L^2(\Omega)} + \|\partial_t^{k-1}\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_{m,1}(1+t)^{-k/2-N/4}, \tag{5.3}$$

in particular,

$$\|\partial_t^{m-1}\nabla u(t)\|_{L^2(\Omega)} \leqslant C d_{m,1}(1+t)^{-m/2-N/4} \tag{5.4}$$

which means case $j = 0$ in assertion (5.2).

Put $w = \partial_t^{m-2} u$, then the function w satisfies $(\Box + \partial_t)w = 0$ with $w|_{\partial\Omega} = 0$, and by Lemma 3.4, we observe that

$$\begin{aligned}\|\nabla w(t)\|_{H^1(\Omega)} &\leqslant C\|\Delta w(t)\|_{L^2(\Omega)} + C\|\nabla w(t)\|_{L^2(\Omega)} \\ &\leqslant C\|\partial_t^2 w(t)\|_{L^2(\Omega)} + C\|\partial_t w(t)\|_{L^2(\Omega)} + C\|\nabla w(t)\|_{L^2(\Omega)}\end{aligned}$$

and hence, from (5.1) and (5.4),

$$\begin{aligned}\|\partial_t^{m-2}\nabla u(t)\|_{H^1(\Omega)} &\leqslant C\|\partial_t^m u(t)\|_{L^2(\Omega)} + C\|\partial_t^{m-1} u(t)\|_{L^2(\Omega)} + C\|\partial_t^{m-2}\nabla u\|_{L^2(\Omega)} \\ &\leqslant C d_{m,1}(1+t)^{-(m-1)/2-N/4},\end{aligned} \tag{5.5}$$

which means case $j = 1$ in assertion (5.2).

Let ℓ be an integer such that $1 \leqslant \ell \leqslant m-1$. We now assume that for $0 \leqslant j \leqslant \ell - 1$,

$$\|\partial_t^{m-j-1}\nabla u\|_{H^j(\Omega)} \leqslant C d_{m,1}(1+t)^{-(m-j)/2-N/4}. \tag{5.6}$$

Put $w = \partial_t^{m-\ell-1} u$ $(j = \ell)$, then the function w satisfies $(\Box + \partial_t)w = 0$ with $w|_{\partial\Omega} = 0$, and by Lemma 3.4, we observe that

$$\begin{aligned}\|\nabla w(t)\|_{H^\ell(\Omega)} &\leqslant C\|\Delta w\|_{H^{\ell-1}(\Omega)} + C\|\nabla w(t)\|_{L^2(\Omega)} \\ &\leqslant C\|\partial_t^2 w(t)\|_{H^{\ell-1}(\Omega)} + C\|\partial_t w(t)\|_{H^{\ell-1}(\Omega)} + C\|\nabla w(t)\|_{L^2(\Omega)},\end{aligned}$$

and hence, from (5.3) and (5.6),

$$\begin{aligned}&\|\partial_t^{m-\ell-1} u(t)\|_{H^\ell(\Omega)} \\ &\quad\leqslant \|\partial_t^{m-\ell+1} u(t)\|_{H^{\ell-1}(\Omega)} + \|\partial_t^{m-\ell} u(t)\|_{H^{\ell-1}(\Omega)} + C\|\partial_t^{m-\ell-1} u(t)\|_{L^2(\Omega)} \\ &\quad\leqslant C d_{m,1}(1+t)^{-(m-\ell)/2-N/4}.\end{aligned}$$

Therefore, the desired estimate (5.2) is valid for $0 \leqslant j \leqslant m-1$. $\Box$

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